# Completion of finding proofs for generalized Langley's problems in elementary geometry (DRAFT20180609)

# 初等幾何で整角四角形を完全制覇

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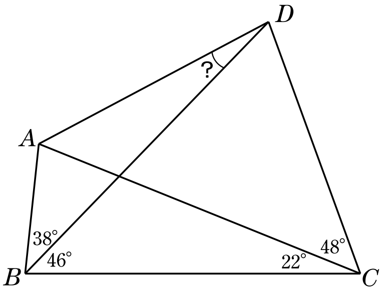
## 1 Groups of problems unsolved in elementary geometry

Formerly, I had been posting problems of "quadrangle with integer angles" (generalized Langley's problem) as the column *"Challenge from Geometry the Great"* (幾何大王からの挑戦状), and in the final post I introduced one of that kind of problem which had not ever been proved by elementary geometry. But finally, a proof for that problem no doubt by elementary geometry was found on October 27th, 2015. Moreover, the method used in that proof can be applied to every generalized Langley's problem including two groups of problems which were considered not having any known proof by elementary geometry until then, and the mission "Find proof by elementary geometry for every generalized Langley's problem", which J. F. Rigby left as a homework in 1978, was completed. The epoch-making proof was presented on the internet by Ms. *aerile_re* (pen name). In this article, I will introduce her method in detail with her consent.

The following is the problem which she solved on Oct. 27th.

**[Q1]**
$ABCD$ is a quadrangle such that $\angle ABD = 38°$, $\angle DBC = 46°$, $\angle BCA = 22°$ and $\angle ACD = 48°$. Prove $\angle BDA = 18°$ by elementary geometry.




A "problem of quadrangle with integer angles" is formulated as follows: In the quadrangle on the left side of Fig.1, angles $a$, $b$, $c$, $d$ are given, and you should find angle $e$ or prove $e$ equals to such value. A quadrangle such that every angle formed by edges and diagonals has integer value in degree (or rational number value in the broad sense) is called a "quadrangle with integer angles". Also, a triangle with a point placed inside of it and connected to each vertexes (like the right side of Fig.1) such that every angle formed by each lines has integer (or rational number) value (sometimes called a "triangle with integer angles") is regarded as that equivalent. In this article, let us call a "problem of quadrangle with integer angles", a "generalized Langley's problem".

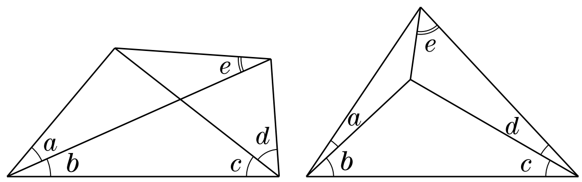
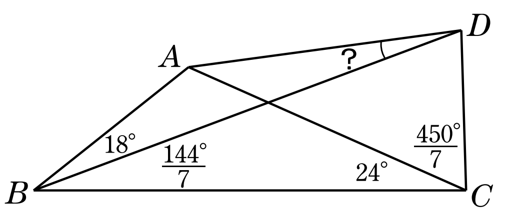



Fig.1

As introduced in the former article, British mathematician J. F. Rigby made systematic research for the proof of the existence of each "quadrangle with integer angles" (in his word "adventitious quadrangle"), which is equivalent to the solution of each generalized Langley's problem, in elementary geometry in 1970's. As a result of the research, almost all problems turned out to be covered by several methods, and only small part of the problems was left unreachable by the systematic path of proof. Let me skip the detail of the systematic proof, one of the problems in the remaining group was Q1.

All non-series adventitious quadrangles, excepting those belong to 1 or 2 parameter series, are divided in 65 small groups, and 8 of them remained unproven by elementary geometry. Gathering equivalent groups from algebraic point of view, these 8 small groups can be categorized into 2 larger groups. If one of the problems in each group is proved by elementary geometry, you can construct a proof for any other problem in the group. It means that we need to prove one more problem in the group other than the group which Q1 belongs to, to complete proofs for all generalized Langley's problem. So I asked Ms. *aerile_re* to prove following Q2, and finally, it was verified no generalized Langley's problem is unprovable in elementary geometry on October 30th, 2015.

**[Q2]**
$ABCD$ is a quadrangle such that $\angle ABD = 18°$, $\angle DBC = \dfrac{144°}{7}$, $\angle BCA = 24°$ and $\angle ACD = \dfrac{450°}{7}$.
Prove $\angle BDA = \dfrac{90°}{7}$ by elementary geometry.




Now, it should be noted that the story that these groups of problems were not proved until then is based on my own recognition. If anyone knows the fact that some proof by elementary geometry for these unsolved problems left by Rigby was presented before Oct. 27th, please let me know.

## 2 Proofs using "3 circumcenter method"

Both of the proofs for Q1 and Q2 by Ms. *aerile_re* are based on “3 circumcenter method” which she invented. First of all, please read these proofs.
(Some expressions in the proofs are changed and complemented. Please refer *http://note.chiebukuro.yahoo.co.jp/detail/n365238* to see the original.)

**Proof for Q1**

Let $P, Q$ be the circumcenters of $\triangle ABC$ and $\triangle DBC$. Let $R$ be the circumcenter of $\triangle QPC$.

Moreover, let $S$ be the point such that $\triangle ASP \equiv \triangle CRP$ and both $B$ and $S$ are on the same side of line $AP$, let $T$ be the point such that $\triangle QTD \equiv \triangle CRQ$ and both $C$ and $T$ are on the same side of line $QD$. Then, $AS = SP = PR = RQ = QT = TD$.

As a result, $\triangle TQR = 60°$ (From here, detailed calculation of angles will be described later), hence $\triangle TQR$ is equilateral.

Let $SPUVW$ be a regular pentagon such that $A$ and it are on the same side of line $SP$, then $\angle UPR = 60°$ and $\triangle UPR$ is equilateral.

Let $DTX$ be an equilateral triangle such that $Q$ and it are on the opposite side of line $DT$, and let $URTY$ be a rhombus. Since $\triangle UYT \equiv \triangle ASW$ and $\triangle YTX \equiv \triangle SWV$, it follows that $\triangle UYX \equiv \triangle ASV$.

Since $DX \mathbin{/\!/} VU$, $VUXD$ is a parallelogram.

Let $S'$ be the point such that $\triangle UYX \equiv \triangle DS'V$, then $\triangle ASV \equiv \triangle DS'V$ and $SADS'$ is an isosceles trapezoid.

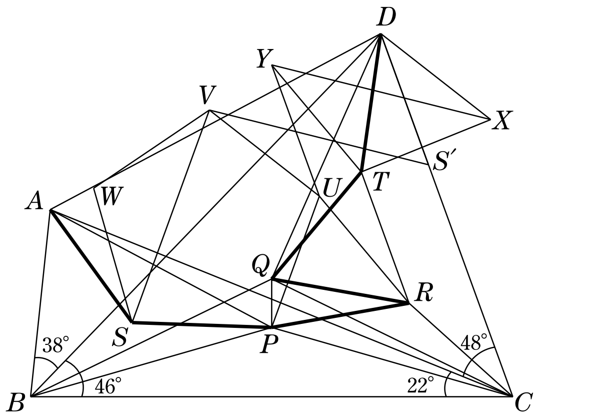



Fig.2

Considering the sum of the interior angles of the heptagon $ASPRQTD$, $\angle SAD + \angle TDA = 136°$.

Since $\angle XDS' = \angle VUY = 32°$, $\angle SAD + \angle S'DA = 164°$. Hence $\angle SAD = 82°$.

It leads to the result that $\angle ADB = 18°$.

**[Calculation of angles]**

$\angle ABC = 84°$, $\angle CAB = 74°$. Therefore
$\angle APC = 168°$, $\angle PCA = \angle CAP = 6°$,
$\angle CPB = 148°$, $\angle PBC = \angle BCP = 16°$.

$\angle DBC = 46°$, $\angle CDB = 64°$. Therefore
$\angle DQC = 92°$, $\angle QCD = \angle CDQ = 44°$,
$\angle CQB = 128°$, $\angle QBC = \angle BCQ = 26°$.

Considering $PB = PC$ and $QB = QC$,
$\angle CQP = 64°$, $\angle QPC = 106°$. Therefore
$\angle CRP = 128°$, $\angle RPC = \angle PCR = 26°$,
$\angle CRQ = 148°$, $\angle RQC = \angle QCR = 16°$, $\angle PRQ = 20°$.

$\angle ASP = \angle CRP = 128°$,
$\angle SPR = \angle APC = 168°$,
$\angle TQR = \angle DQC - 2\angle RQC = 60°$,
$\angle QTD = \angle CRQ = 148°$,
$\angle UPR = \angle SPR - 108° = 60°$,
$\angle ASW = \angle ASP - 108° = 20°$,
$\angle UYT = \angle URT = \angle PRQ = 20°$,
$\angle RTY = \angle YUR = 160°$,
$\angle YTX = \angle QTD + 120° - \angle RTY = 108°$.

Let lines $DX$ and $TR$ intersect at point $Z$, then
$\angle XZR = \angle XTR - 60° = 32°$,

$\angle VUY = 360° - 108° - 60° - \angle \mathrm{YUR} = 32°$.
Since $TR \parallel YU$, $DX \parallel VU$.

According to the main part of the proof, $\angle SAD = 82°$.
Hence, $\angle DAC = \angle SAD - 26° - 6° = 50°$,
$\angle BDA = \angle DBC + \angle BCA - \angle DAC = 18°$.

**Proof for Q2**

Let $P, Q$ be the circumcenters of $\triangle ABC$ and $\triangle DBC$. Let $R$ be the circumcenter of $\triangle BPQ$.

Moreover, let $S$ be the point such that $\triangle ASP \equiv \triangle PRB$ and both $B$ and $S$ are on the same side of line $AP$, let $T$ be the point such that $\triangle QTD \equiv \triangle BRQ$ and both $C$ and $T$ are on the same side of line $QD$. Then, $AS = SP = PR = RQ = QT = TD$.

Let $PR_3RR_4R_2$ be a regular pentagon such that points $Q$ and $R_3$ are on the same side of line $PR$, then $PRR_2R_3R_4$ is a regular star pentagon. Since $\angle TQR = \angle QRR_2 = \dfrac{900°}{7}$, we can make the regular heptagon $TQRR_2WVU$.

Here, let $l$ be line $R_4U$. According to the symmetry of the regular pentagon and the regular heptagon, $T$, $Q$, $R$, $R_3$ are symmetric to $V$, $W$, $R_2$, $P$, respectively, through $l$. Further, let $A'$, $D'$, $S'$ be the points symmetric to $A$, $D$, $S$, respectively, through $l$.

Moreover, let $XR_4R_3$ and $YR_4P$ be equilateral triangles such that $P$ and $X$ are on the opposite side of line $R_4R_3$ and $R_3$ and $Y$ are on the opposite side of line $R_4P$, then $X$ and $Y$ are symmetric through $l$.

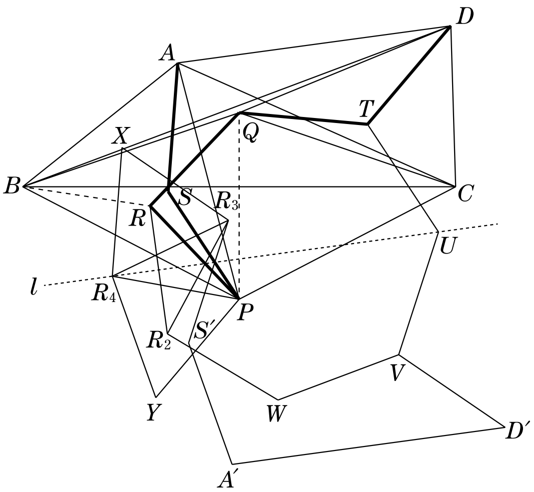


Fig.3

Since $AS \parallel XR_4$, $A'S' \parallel YR_4$, we can make polygonal line $AK_1K_2K_3A'$ which shares start and end points with polygonal line $ASPYR_4XR_3S'A'$ such that $AK_1 \parallel PY$, $K_1K_2 \parallel SP$, $K_2K_3 \parallel R_3S'$ and $K_3A' \parallel XR_3$. Furthermore, it matches polygonal line $DTUVD'$ by parallel translation (Detail will be described later), hence $AD \parallel A'D'$. Moreover $AD$ and $A'D'$ are symmetric through $l$, hence $AD \parallel l$. It leads to the conclusion that $\angle BDA = \dfrac{90°}{7}$.

**[Calculation of angles]**

$\angle BCA = 24°$, $\angle CAB = \dfrac{822°}{7}$. Therefore
$\angle BPA = 48°$, $\angle PAB = \angle ABP = 66°$,
$\angle BPC = \dfrac{876°}{7}$, $\angle PCB = \angle CBP = \dfrac{192°}{7}$.

$\angle BCD = \dfrac{618°}{7}$, $\angle CDB = \dfrac{498°}{7}$. Therefore
$\angle BQD = \dfrac{1236°}{7}$, $\angle QDB = \angle DBQ = \dfrac{12°}{7}$,
$\angle CQB = \dfrac{996°}{7}$, $\angle QBC = \angle BCQ = \dfrac{132°}{7}$.
Considering $PB = PC$ and $QB = QC$,
$\angle PQB = \dfrac{498°}{7}$, $\angle BPQ = \dfrac{438°}{7}$. Therefore
$\angle PRB = \dfrac{996°}{7}$, $\angle RBP = \angle BPR = \dfrac{132°}{7}$,
$\angle BRQ = \dfrac{876°}{7}$, $\angle RQB = \angle QBR = \dfrac{192°}{7}$,
$\angle QRP = \dfrac{648°}{7}$.

$\angle ASP = \angle PRB = \dfrac{996°}{7}$,
$\angle RPS = \angle BPA - 2\angle BPR = \dfrac{72°}{7}$,
$\angle TQR = 360° - \angle BQD - 2\angle RQB = \dfrac{900°}{7}$,
$\angle QTD = \angle BRQ = \dfrac{876°}{7}$.

$\angle QRR_2 = \angle QRP + 36° = \dfrac{900°}{7}$,
$\angle R_4PS = \angle RPS + 36° = \dfrac{324°}{7}$.
Let lines $SP$ and $l$ intersect at point $Z_1$,
then $\angle R_4Z_1S = \angle R_4PS + 18° = \dfrac{450°}{7}$,
$\angle R_4UT = \dfrac{450°}{7} = \angle \mathrm{R}_4Z_1S$, hence $TU \parallel SP$.

Since $\angle YPS = \angle R_4PS + 60° = \dfrac{744°}{7}$ and
$\angle DTU = 360° - \angle QTD - \dfrac{900°}{7} = \dfrac{744°}{7}$,
$DT \parallel PY$. Similarly, $VU \parallel S'R_3$, $D'V \parallel R_3X$.

Let lines $AS$ and $l$ intersect at point $Z_2$, then
$\angle AZ_2U = \angle ASP - \angle R_4Z_1S = 78°$,
$\angle XR_4U = 60° + 18° = 78° = \angle AZ_2U$, hence
$AS \parallel XR_4$. Similarly, $A'S' \parallel YR_4$.

According to the main part of the proof, $AD \parallel l$.
Hence, $\angle TDA = \angle DTU - \angle R_4UT = \dfrac{294°}{7}$,

$$\angle BDA = \angle TDA - \angle TDQ - \angle QDB = \frac{90^\circ}{7}.$$

**[Operating polygonal lines]**

Let $XR_4YK_2$, $ASPK_4$ and $R_3S'A'K_5$ be rhombuses, then $K_4P \mathbin{/\!/} AS \mathbin{/\!/} XR_4 \mathbin{/\!/} K_2Y$, $K_5R_3 \mathbin{/\!/} A'S' \mathbin{/\!/} YR_4 \mathbin{/\!/} K_2X$, hence, quadrilaterals $K_4K_2YP$ and $XK_2K_5R_3$ are also rhombuses.

Moreover, let $AK_1K_2K_4$ and $K_2K_3A'K_5$ be rhombuses, then $AK_1 \mathbin{/\!/} K_4K_2 \mathbin{/\!/} PY \mathbin{/\!/} DT$, $K_1K_2 \mathbin{/\!/} AK_4 \mathbin{/\!/} SP \mathbin{/\!/} TU$. Similarly $K_2K_3 \mathbin{/\!/} R_3S' \mathbin{/\!/} UV$, $K_3A' \mathbin{/\!/} XR_3 \mathbin{/\!/} VD'$.

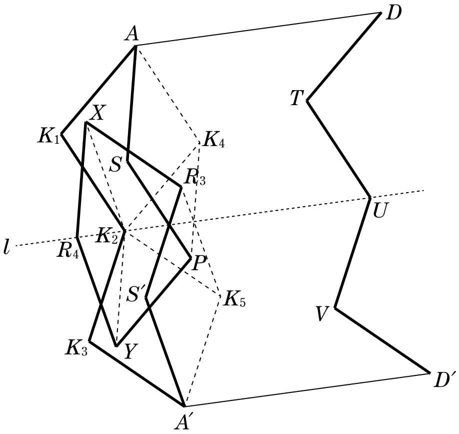


Fig.4

Actually, it is clear that $K_2$ lies on $l$, so the fact that $AD \mathbin{/\!/} l$ can be shown without using points $A'$, $D'$, $S'$, $K_3$ and $K_5$.

# 3 Make polygonal line using 3 circumcenters

The main idea of Ms. *aerile_re*'s "3 circumcenter method" is condensed into first few lines of each of the two proofs.

A generalized Langley's problem can be considered as a problem about two triangles $ABC$ and $DBC$ which share the base and every interior angles of those are known, to find the direction of $AD$ seen from the base (to be accurate, the angle of rotation from $\overrightarrow{BC}$ to $\overrightarrow{AD}$). In case that $D$ lies inside $\triangle ABC$, it's a problem of "triangle with integer angles". Let us take $Q$ as the circumcenter of $\triangle ABC$, and $P$ as the circumcenter of $\triangle DBC$. Then both $P$ and $Q$ lie on the perpendicular bisector of segment $BC$, and every interior angles of $\triangle BPQ$ and $\triangle CPQ$ are known. And furthermore, taking $R$ as a circumcenter of $\triangle CPQ$ and taking $S$ and $T$ such that $\triangle ASP \equiv \triangle CRP$ and $\triangle QTD \equiv \triangle CRQ$, enable us to connect $A$ and $D$ by a polygonal line $ASPRQTD$ such that every line segment composing the polygonal line has the same length and known angle relative to $BC$. As a result, we can replace the problem itself with a problem to find the relative direction from the start point to the end point of the polygonal line.

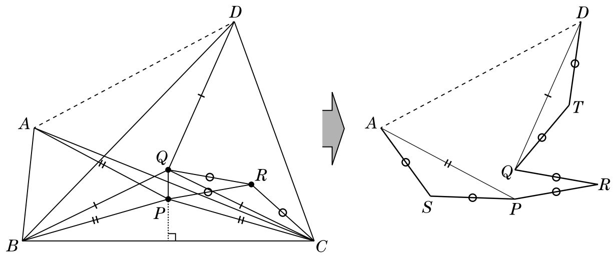


Fig.5

Here, you can make $\triangle ASP$ and $\triangle QTD$ on eiher side of lines $AP$ and $QD$. Moreover, you can take $R$ as a circumcenter of $\triangle BPQ$ instead of $\triangle CPQ$. In that case, you should make triangles congruent with $\triangle BRP$ and $\triangle BRQ$ instead of those congruent with $\triangle CRP$ and $\triangle CRQ$. These replacement only cause some change of the appearing order of segments, but the relative angles of each segments do not change.

This problem to find the direction from the start point to the end point of a polygonal line composed of segments of the same length, can be considered as a problem to find the argument of sum of 6 $n$th root of unity. And the existence of the segments which correspond to each complex numbers lets the geometrical consideration on the Gaussian plane be translated directly into the proof in elementary geometry.

# 4 Equivalent transformation of the polygonal line in elementary geometry

Setting the algebraic calculation in the background aside, let me show the specific methods to construct a proof for the problem to find the direction from the start point to the end point of a polygonal line composed of equal segments. The basic concept is to transform the polygonal line into the figure which we can easily find the direction from the start point to the end point of it, only using some operations which do not change the start point and the end point. These operations are following three; 'rearrangement', 'replacement' and 'cancellation'.

'Rearrangement' corresponds to the change of calculating order in the case that the polygonal line is considered as the sum of vectors. Using parallelograms (in this case of equal segments, rhombuses) is an easy way to implement it in elementary geometry. Using rhombus $ABCD$, polygonal line $ABC$ can be replaced with polygonal line $ADC$. Repeating such operations, we can make a polygonal line composed of the same set of segments (segments with same direction and same length are considered identical here) in any order.

'Replacement' means replacing a part of the polygonal line with another polygonal line, and a regular polygon is used for this operation. If a part of the polygonal line can be regarded as a part of a regular polygon, it can be replaced with the rest path of the regular polygon. The method used in 'rearrangement' can also be considered as 'replacement' using a rhombus.

'Cancellation' is to delete a pair of segments with opposite directions from the polygonal line. The path can be shortened by rearranging segments so that the pair adjoin each other, or by making parallel displacement of the path between the pair.

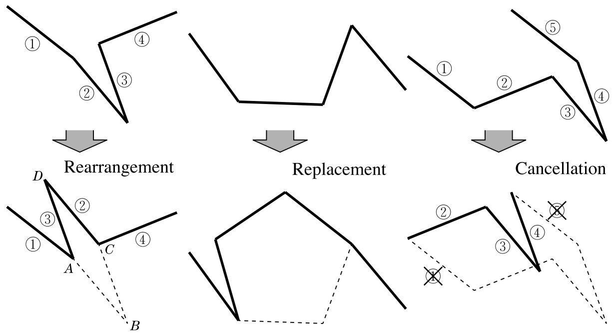



Fig.6

The goal of the transformation of the polygonal line $ASPRQTD$ using these method is the figure where the direction of segment $AD$ is known. For that purpose, the polygonal line should be transformed into a part of a line-symmetric figure. In regard to this, different ideas are used in the proofs for Q1 and Q2. In the proof of Q1, points $A$ and $D$ are symmetric with respect to the axis of symmetry. On the other hand in the proof of Q2, line $AD$ itself is the axis of symmetry of the figure, in principle. (In the above-mentioned proof of Q2, the line-symmetric figure is drawn at the other place than $AD$ to reduce the procedure of rearrangement.)

The process of the proofs for Q1 and Q2 focusing on the transformation of the polygonal line is shown in Fig.7 and 8.

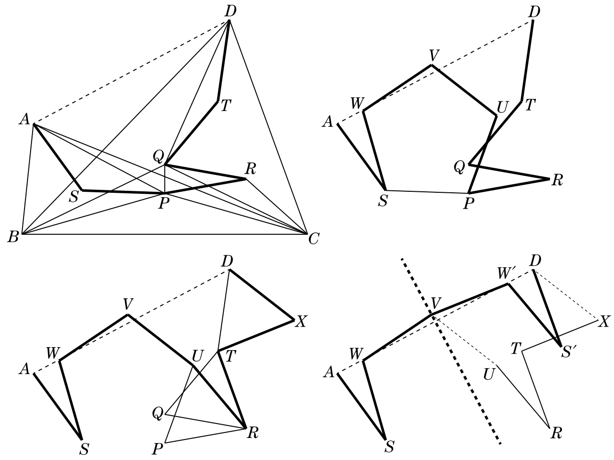


Fig.7

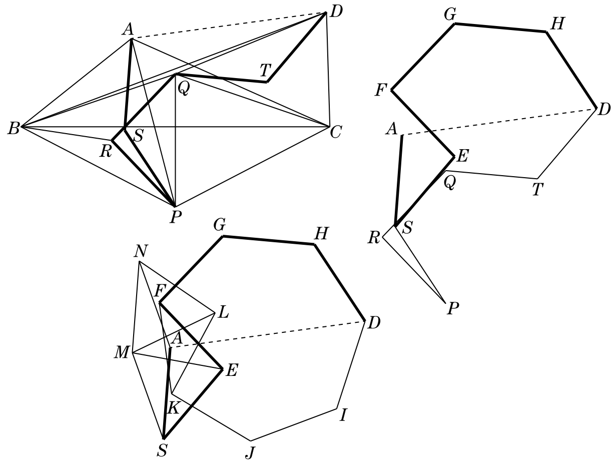


Fig.8

Regarding Q1, $SP$ is replaced with $SWVUP$ using regular pentagon at first, and further replacement using three equilateral triangles produces polygonal line $ASWVURTXD$. After the cancellation of $VU$ and $XD$ and some rearrangement, finally appeared polygonal line $ASWVW'S'D$ is line-symmetric with respect to the bisector of $\angle WVW'$, and $AD$ is perpendicular to the axis of symmetry.

Regarding Q2, polygonal line $ASPRQTD$ is rearranged into $ASEFGHD$ at first. After drawing regular heptagon $FGHDIJK$, regular star pentagon $EFKLM$, equilateral triangles $SEM$ and $MLN$, and rhombus $NMSA$, it can be confirmed that point $A$ lies on line $MD$ which is the axis of symmetry for all drawn figures.

It should be noticed that these two methods can be applied to any generalized Langley's problem using algebraic calculation described next.

## 5 Geometry on the Gaussian plane

In order to construct the proofs for Q1 and Q2, Ms. *aerile_re* executed following calculations using $n$th root of unity, and translated the result into the relationship within plane figures. (Here, some expressions are also changed from the original.)

**[Background calculations for Q1 (Method 1)]**

The angles of each directed segments in polygonal line $ASPRQTD$ relative to $BC$ are $-54°$, $-2°$, $10°$, $170°$, $50°$ and $82°$ in order, and the goal of the proof is to show the relative angle of $AD$ equals to $28°$. Let $z = e^{\frac{2\pi i}{360}}$, then showing that

$$f(z) = z^{-54} + z^{-2} + z^{10} + z^{170} + z^{50} + z^{82}$$

is a multiplication of $z^{28}$ by real number is enough for the purpose.

The minimum polynomial of $z$ is the 360th cyclotomic polynomial $F_{360}(x)$ whose degree is $\phi(360) = 96$. Therefore, we can transform $f(z)$ such that the degree of each term of it is within the range from $28 - 48$ to $28 + 47$, uniquely. Let the transformed polynomial be $g(z)$, then $z^{20}g(z)$ should be the remainder obtained by dividing

$$z^{20}f(z) = z^{326} + z^{18} + z^{30} + z^{190} + z^{70} + z^{102} \quad \text{by}$$

$$F_{360}(z) = z^{96} + z^{84} - z^{60} - z^{48} - z^{36} + z^{12} + 1.$$

Using the fact, it can be calculated that

$$g(z) = z^{28}(-z^{42} - z^{38} + z^{22} + z^{18} + z^{6} + z^{-6} + z^{-18} + z^{-22} - z^{-38} - z^{-42}),$$

and this symmetric form means that $f(z) = g(z)$ is a multiplication of $z^{28}$ by real number. Using the equalities

$$-z^{42} + z^{-18} = z^{-78}, \quad z^{18} - z^{-42} = z^{78},$$
$$-z^{38} + z^{-22} = z^{-82}, \quad z^{22} - z^{-38} = z^{82},$$

we can make further transformation into a polynomial with less terms like

$$g_0(z) = z^{28}(z^{82} + z^{78} + z^{6} + z^{-6} + z^{-78} + z^{-82}).$$

Now, transforming the polygonal line representing $f(z)$ into the polygonal line representing $g_0(z)$ forms the proof by elementary geometry.

The relation between the polynomials,

$$\begin{aligned} &f(z) - g_0(z) \\ &= f(z) + z^{208}(z^{82} + z^{78} + z^{6} + z^{-6} + z^{-78} + z^{-82}) \\ &= z^{10} + z^{50} + z^{82} + z^{126} + z^{130} + z^{170} \\ &\quad + z^{202} + z^{214} + z^{286} + z^{290} + z^{306} + z^{358} = 0, \end{aligned}$$

can be described using Fig.9 in which the terms are plotted on the Gaussian plane, by the fact that the sum of the complex numbers located on the vertexes of each regular pentagon or equilateral triangle in Fig.9 is equal to 0. Here, the black dots are the terms of $f(z)$, the white dots are the terms of $-g_0(z)$, and the gray dots are canceled as pairs and do not appear as terms. The 'replacements' used in the proof for Q1 correspond to the regular pentagons and the equilateral triangles in this figure. (Note that each segment in these polygonal line corresponds to a dot here.)

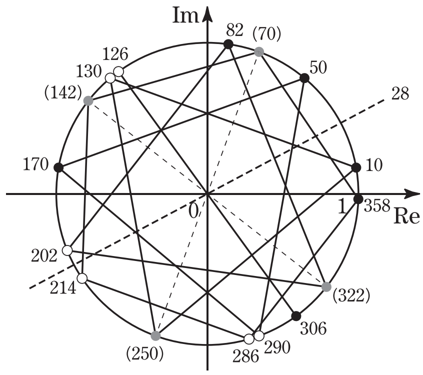


Fig.9

**[Background calculations for Q2 (Method 2)]**

Let $t = \dfrac{6°}{7}$. The angles of each directed segments in polygonal line $ASPRQTD$ relative to $BC$ are $-110t$, $-66t$, $156t$, $54t$, $-6t$ and $58t$, and the goal of the proof is to show the relative angle of $AD$ equals to $9t$. Let $z = e^{\frac{2\pi i}{420}}$, then showing the fact that

$$f(z) = z^{-110} + z^{-66} + z^{156} + z^{54} + z^{-6} + z^{58}$$

is a multiplication of $z^{9}$ by real number, which is equivalent to the fact that $z^{-9}f(z)$ is a real number, is enough for the purpose.

The relation

$$\begin{aligned} &z^{-9}f(z) - \overline{z^{-9}f(z)} \\ &= z^{301} + z^{345} + z^{147} + z^{45} + z^{405} + z^{49} \\ &\quad + z^{329} + z^{285} + z^{63} + z^{165} + z^{225} + z^{161} = 0 \end{aligned}$$

is clearly shown in Fig.10. (The black dots are the terms of $z^{-9}f(z)$, the white dots are the terms of $-\overline{z^{-9}f(z)}$, and the gray dots are canceled as pairs.) A regular heptagon, a regular pentagon, and 2 equilateral triangles in this figure correspond to those in Fig.8. The reason why a regular star pentagon is used in Fig.8 is that $z^{147}$ which corresponds to $PR$ ($EF$) and $z^{315}$ which corresponds to $FK$ are not next to each other on the regular pentagon in Fig.10.

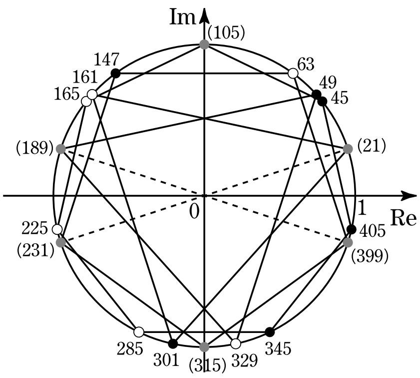



Fig.10

This time, two different methods are used for these proofs, but actually, the second method is much easier in calculation and in finding graphical relationships on the Gaussian plane. Combining "3 circumcenter method" with the calculation by this second method can almost mechanically construct the proof of arbitrary generalized Langley's problem by elementary geometry.

# 6 The correspondence to the conventional proof

The calculation by the second method completely corresponds to the algebraic calculation about three-diagonal intersections of regular $n$-gons which is equivalent to quadrangles with integer angles, used in the conventional proof for the problem like Q1 and Q2 which does not depend on the elementary geometry.

Poonen and Rubinstein treated the existence condition of three-diagonal intersections of regular $n$-gons as follows.

Let the circumference of a circle cut into six arcs with lengths in integer ratio, and let $U$, $X$, $V$, $Y$, $W$, and $Z$ denote the ratio of each arc in order when the entire circumference is 1. According to the trigonometric form of Ceva's theorem, it is equivalent to

$$\sin \pi U \sin \pi V \sin \pi W = \sin \pi X \sin \pi Y \sin \pi Z$$

that those three chords connecting each division point as shown in Fig.11 intersect at a single point. Moreover, using the fact that $\sin \theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, the condition can be arranged like follows.

$$\sum_{j=1}^{6} e^{i\pi\alpha_j} + \sum_{j=1}^{6} e^{-i\pi\alpha_j} = 0, \quad \text{where}$$

$$\begin{aligned}
\alpha_1 &= V + W - U - 1/2,\\
\alpha_2 &= W + U - V - 1/2,\\
\alpha_3 &= U + V - W - 1/2,\\
\alpha_4 &= Y + Z - X + 1/2,\\
\alpha_5 &= Z + X - Y + 1/2,\\
\alpha_6 &= X + Y - Z + 1/2
\end{aligned}$$

For the sake of later argument, let the formula be changed like

$$\sum_{j=1}^{6} e^{i\beta_j} - \sum_{j=1}^{6} e^{-i\beta_j} = 0,$$

$$\beta_j = \pi\alpha_j + \pi/2 \quad (j = 1, \cdots, 6).$$

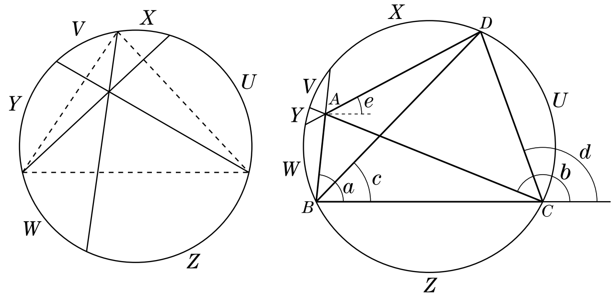



Fig.11

Let $ABCD$ be a quadrangle with integer angles such that point $A$ is located inside the circumscribed circle of $\triangle DBC$, and let $a$, $b$, $c$, $d$, and $e$ denote the angles of directed segments $BA$, $CA$, $BD$, $CD$, and $AD$ relative to $BC$. Now, consider point $A$ as a three-diagonal intersection of an inscribed regular $n$-gon in the circumscribed circle of $\triangle DBC$ like the right side of Fig.11, then

$$\pi U = c, \quad \pi V = -a + b + c - d, \quad \pi W = c - e,$$
$$\pi X = a - c, \quad \pi Y = -b - c + e + \pi, \quad \pi Z = -c + d,$$

and as a result,

$$\begin{aligned}
\beta_1 &= -a + b + c - d - e,\\
\beta_2 &= a - b + c + d - e,\\
\beta_3 &= -a + b + c - d + e,\\
\beta_4 &= -a - b - c + d + e + 2\pi,\\
\beta_5 &= a + b - c + d - e,\\
\beta_6 &= a - b - c - d + e + 2\pi.
\end{aligned}$$

On the other hand, when "3 circumcenter method" is applied to this quadrangle with integer angles, the angles of each directed segments in generated polygonal line relative to $AD$ are as follows in order.

$$\begin{aligned}
\theta_1 &= a + b - c + d - e,\\
\theta_2 &= a + b + c - d - e + \pi,\\
\theta_3 &= -a + b + c - d - e,\\
\theta_4 &= a - b - c + d - e + \pi,\\
\theta_5 &= -a + b + c + d - e + \pi,\\
\theta_6 &= a - b + c + d - e.
\end{aligned}$$

Now, it can be easily checked that the terms appearing in the formula

$$\sum_{j=1}^{6} e^{i\theta_j} - \sum_{j=1}^{6} e^{-i\theta_j} = \sum_{j=1}^{6} e^{i\theta_j} + \sum_{j=1}^{6} e^{i(\pi-\theta_j)} = 0$$

which shows that the angle of the segment connecting the start point and the end point of the polygonal line relative to $AD$ is equal to 0 using the second calculating method, completely match those in the formula

$$\sum_{j=1}^{6} e^{i\beta_j} - \sum_{j=1}^{6} e^{-i\beta_j} = \sum_{j=1}^{6} e^{i\beta_j} + \sum_{j=1}^{6} e^{i(\pi-\beta_j)} = 0.$$

Poonen and Rubinstein successfully made the complete list of three-diagonal intersections of regular polygons by classifying the combination of the regular polygons on the Gaussian plane drawn inevitably by plotting the 12 $n$th roots of unity with 0-sum obtained from these intersections, and this “3 circumcenter method” is the key to let the clear relationship on the Gaussian plane which has been unable to be reflected in the proof by elementary geometry until now, appear directly in the figure for the proof.

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